Numerical Methods & Computer Aided Engineering

with MATLAB & ANSYS

Computer Aid Engineering (CAE) is an important tool widely used for designing and analyzing engineering problems nowadays. CAE requires profound backgrounds in mathematics and numerical methods. This book presents clear explanations of fundamental theories behind CAE and wide ranges of applications. The book contains 14 chapters with essential materials that are taught in the CAE course. The finite difference method is presented to solve the boundary and initial value problems. The finite element method is explained to analyze problems governed by the elliptic, parabolic and hyperbolic equations. Associated computer codes are also developed using key features in MATLAB to demonstrate underlying computational processes in CAE programs. ANSYS Software procedures for analyzing the heat transfer, structural and fluid flow problems are illustrated, step by step, in details.

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- Presentation is easy to understand using simple explanations and equations along with illustrations.
- Materials are clear and well-organized in an easy-to-follow approach with logical progression through mathematics and computational methods.
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- Lots of exercises to practice and accelerate understandings at the end of each chapter.

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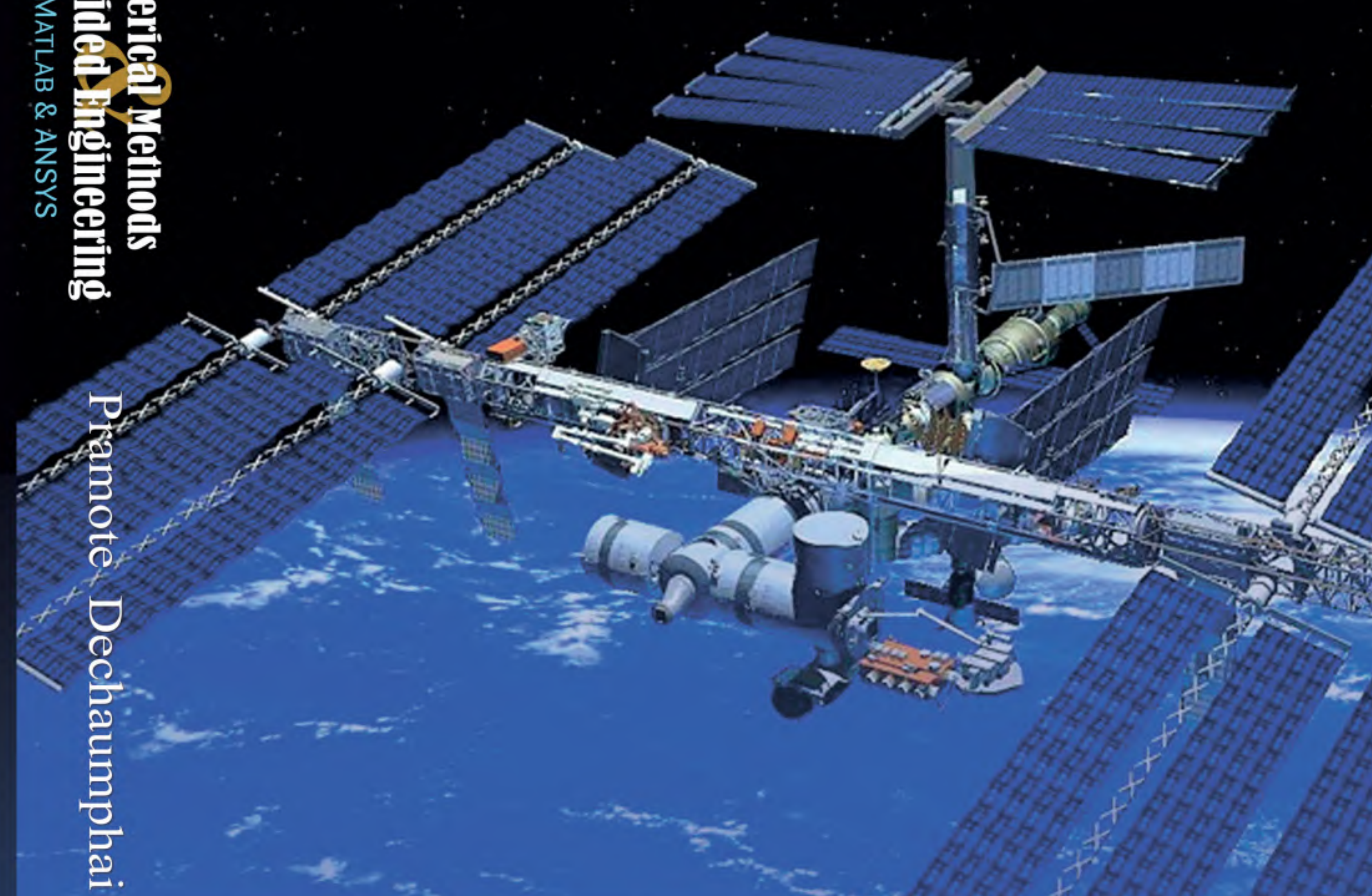

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Chapter 6

Computer Software

6.1 Introduction

Scientific and engineering problems are characterized by differential equations that describe the underlying physical behavior. As demonstrated in the preceding chapters, the finite element equations corresponding to these differential equations can be derived, resulting in a set of algebraic equations for a typical finite element. Since the problem model is discretized into elements, different sets of algebraic equations are derived and assembled to form a large set of algebraic equations. The initial and boundary conditions of the problem are then imposed before solving for the solutions at all nodes in the model.

The aforementioned solving routine has led to the development of various finite element software packages. To analyze a specific problem, the user simply inputs the governing differential equations into the software. The software automatically generates the corresponding finite element equations for the elements. The finite element process, as described above, is then performed to yield the final solutions. These solutions are conveniently displayed as color graphics on the computer screen.

In this chapter, we will utilize the FEATool Multiphysics and MATLAB PDE Toolbox software to analyze the three fundamental classes of differential equations: elliptic, parabolic, and hyperbolic equations. Both of the software are user-friendly and suitable for learning the finite element method. Detailed instructions on how to use these software are provided in Appendices.

6.2 One-Dimensional Elliptic Problem

6.2.1 Differential Equation

The elliptic equation in one dimension is in the form of ordinary differential equation,

$$-\frac{d}{dx}\left(c\frac{du}{dx}\right) + a\,u \;=\; f \qquad (6.1)$$

where c, a and f are constants or may be function of x. The unknown in the above equation is the dependent variable $u(x)$ that depends on x coordinate.

The ordinary differential equation above is solved together with the boundary conditions at both ends of the domain length for the solution of $u(x)$. The problem statement is described in Fig. 6.1. If the boundary conditions are simple, an exact solution to the problem can be derived in closed-form expression.

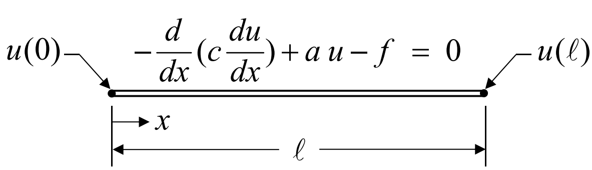


Figure 6.1 One-dimensional elliptic problem.

6.2.2 Element Equations and Matrices

By using the method of weighted residuals as explained in chapter 4, the finite element equations corresponding to the differential equation, Eq. (6.1) can be derived. For simplicity, we use the 2-node element with linear interpolations as shown in Fig. 6.2 to solve this problem.

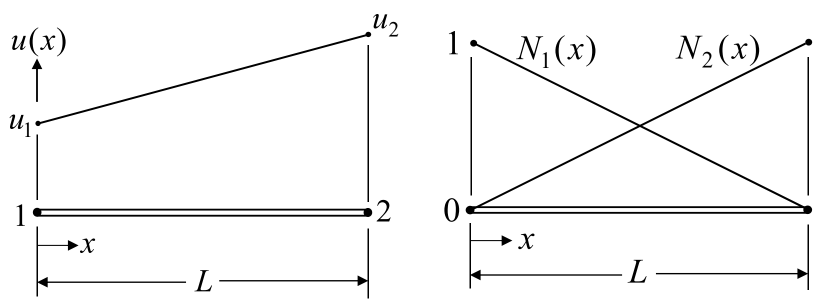


Figure 6.2 Two-node element with linear interpolations.

The distribution of $u(x)$ along the element length L is,

$$\begin{aligned} u(x) &= \left(1-\frac{x}{L}\right)u_1 + \left(\frac{x}{L}\right)u_2 &&= N_1u_1 + N_2u_2 \\ &= \lfloor N_1 \;\; N_2 \rfloor \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} &&= \underset{(1\times2)}{\lfloor N(x) \rfloor} \underset{(2\times1)}{\{u\}} \end{aligned} \tag{6.2}$$

The derived finite element equations are in the form,

$$\underset{(2\times2)}{[K]}\underset{(2\times1)}{\{u\}} + \underset{(2\times2)}{[H]}\underset{(2\times1)}{\{u\}} = \underset{(2\times1)}{\{Q\}} + \underset{(2\times1)}{\{F\}} \tag{6.3}$$

where,

$$\underset{(2\times2)}{[K]} = \int_0^L c \underset{(2\times1)}{\left\{\frac{dN}{dx}\right\}} \underset{(1\times2)}{\left\lfloor \frac{dN}{dx} \right\rfloor} dx \tag{6.4a}$$

$$\underset{(2\times2)}{[H]} = \int_0^L a \underset{(2\times1)}{\{N\}} \underset{(1\times2)}{\lfloor N \rfloor} dx \tag{6.4b}$$

$$\underset{(2\times1)}{\{Q\}} = \left(\underset{(2\times1)}{\{N\}} c \frac{du}{dx} \right) \Bigg|_0^L \tag{6.4c}$$

$$\underset{(2\times1)}{\{F\}} = \int_0^L \underset{(2\times1)}{\{N\}} f dx \tag{6.4d}$$

If the coefficients c, a and f are constant, the above element matrices are,

$$[K] = \frac{c}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad : \quad [H] = \frac{aL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\{Q\} = \begin{Bmatrix} -c\dfrac{du}{dx}(0) \\ c\dfrac{du}{dx}(L) \end{Bmatrix} \quad : \quad \{F\} = \frac{fL}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{6.5}$$

The finite element equations are thus in the form,

$$\frac{c}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \frac{aL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} -c\dfrac{du}{dx}(0) \\ c\dfrac{du}{dx}(L) \end{Bmatrix} + \frac{fL}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{6.6}$$

The finite element equations above will be used to determine the approximate solution as shown by an example in the next section.

6.2.3 Example

Use two linear finite elements to determine the approximate solution of the ordinary differential equation,

$$-\frac{d^2u}{dx^2} - 4u = 4 \tag{6.7}$$

in the domain of $0 \le x \le 1$ with the boundary conditions of $u(0) = 0$ and $u(1) = 1$. Plot to compare the approximate solution with the exact solution of,

$$\bar{u}(x) = \cos(2x) + \frac{2 - \cos(2)}{\sin(2)}\sin(2x) - 1 \tag{6.8}$$

By comparing the differential equation of this example with the differential equation in the general form, Eq. (6.1), the coefficients are $c = 1$, $a = -4$ and $f = 4$.

We start by creating a mesh containing 2 elements with 3 nodes as shown in Fig. 6.3, Herein, we will use the FEATool software to solve for the approximate solutions. The toolbox uses the finite element method with one-dimensional elements to discretize the domain length. We will also demonstrate the accuracy of the finite element solutions obtained from the toolbox by comparing them with the exact solution.

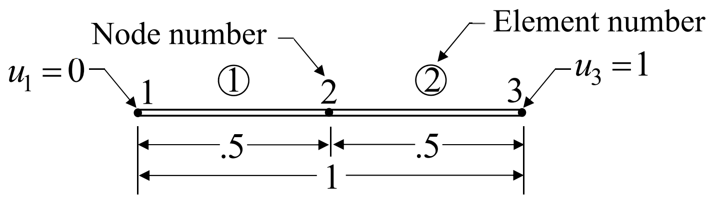


Figure 6.3 Model with two-node linear elements.

To solve this problem by using the FEATool software, we first click at the File item and then proceed with the New Model option, the New Model window will appear. In this New Model Window, we select `1D` under the `Select Space Dimensions`, and choose `Custom Equation` under the `Select Physics`, then click the OK button. Next, we click at the Geometry button and then proceed to select the Create Line button beneath it, the small Create Line window will pop up. We enter 0 and 1 as x min and x max, respectively and click OK button, then the line L1 will appear on the working screen.

Next, we click at the `Grid` button, change the Grid Size to 0.5, and click at the Generate button, the finite element model with 2 elements and 3 nodes will appear. Then, we click at the `Equation` button, the Equation Settings window will appear. On this window, we click at the `edit` button and change the equation to – (ux_x) – 4*x = 4, and click OK. We continue by clicking the Apply and OK button in the Equation Settings window, respectively. Note that, the special notations in the FEATool software, ux denotes the first derivative of du/dx, while ux_x denotes the second derivative of d^2u/dx^2.

The two boundary conditions at both ends of the model can now be applied. By selecting the `Boundary` button, the Boundary Settings window will appear. Select 1 (left-end grid) under Boundaries Item and enter 0 as the Boundary Coefficients for the Dirichlet condition. Similarly, select 2 (right-end grid) under Boundaries and enter 1 as the Boundary Coefficients for the same Dirichlet condition, Then, click at the Apply and OK button, respectively.

Finally, we can solve the problem by selecting the `Solve` and the = (Equal sign) button, respectively, the analysis will be performed. The finite element solutions of $u(x)$ that varies with x for each element will be displayed automatically on the computer screen as shown in Fig. 6.4.

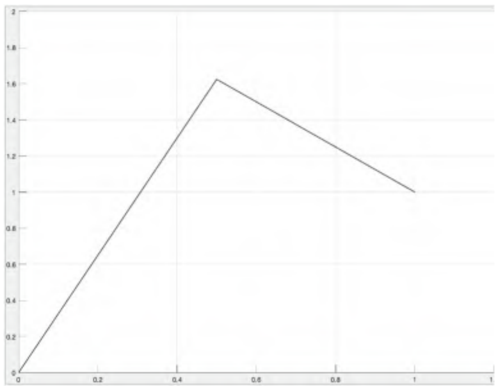


Figure 6.4 Finite element solution of $u(x)$ from FEATool.

By discretizing the domain length with only 2 elements and 3 nodes as shown in Fig. 6.3, the FEATool gives the solution of u_2 = 1.625 at node 2 which is seen in Fig. 6.4. The exact solution at this node (x = 0.5) is 1.776 for which the error is 8.5%. Distribution of this finite element solution is compared with the exact solution, Eq. (6.8), as shown in Fig. 6.5.

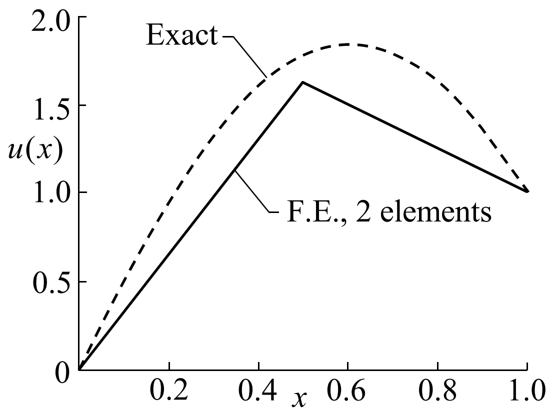


Figure 6.5 Comparative finite element and exact solutions.

The finite element model is further refined by using 4 and 10 elements along the domain length, respectively. The computed solutions from the three finite element meshes are compared with the exact solution as shown in Fig. 6.6. The figure demonstrates that the finite element solutions approach the exact solution as the meshes are refined.

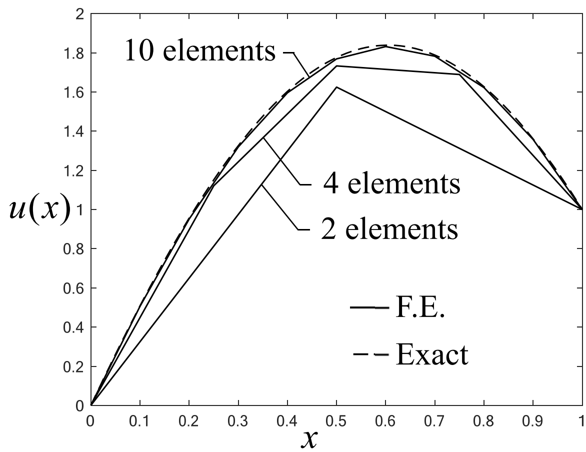


Figure 6.6 Convergence of the finite element solutions.

6.3 Two-Dimensional Elliptic Problem

6.3.1 Differential Equation

The elliptic partial differential equation in two dimensions is in the form of,

$$-\left(\frac{\partial}{\partial x}\left(c\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(c\frac{\partial u}{\partial y}\right)\right) + a\,u \;=\; f \qquad (6.9)$$

where the unknown u is function of x- and y-coordinates. The coefficients c, a and f may be constant or function of x and y.

The domain geometry could be arbitrary as shown in the Fig. 6.7.

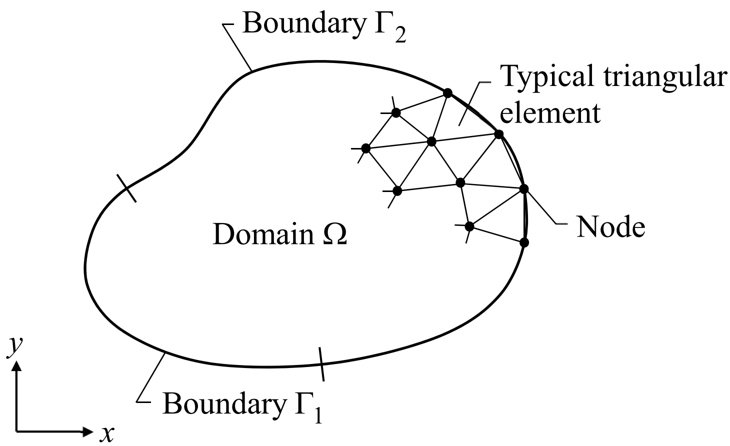


Figure 6.7 Domain discretized into elements.

The elliptic differential equation above is solved together with the boundary conditions which may consist of,

(a) *Dirichlet condition.* The value of u is specified along the boundary Γ_1 as,

$$hu = r \tag{6.10a}$$

where h and r may be constant or function of x and y.

(b) *Neumann condition.* The derivative of u is specified along the boundary Γ_2 as,

$$c\frac{\partial u}{\partial n} + qu = g \tag{6.10b}$$

where c, q and g may be constant or function of x and y. The symbol n denotes the unit vector normal to the boundary.

6.3.2 Element Equations and Matrices

In this section, we will first use the MATLAB PDE Toolbox to analyze the problems. The toolbox always discretizes the two-dimensional domain into a number of the three-node triangular elements. The nodal unknowns at the three nodes are denoted by u_1, u_2 and u_3, respectively. Distribution of $u(x,y)$ over the element is assumed in the form,

$$u(x,y) = N_1(x,y)u_1 + N_2(x,y)u_2 + N_3(x,y)u_3 \qquad (6.11)$$

where N_i, i = 1, 2, 3 are the element interpolation functions which depend on the element area A and the three nodal coordinates x_i and y_i as shown in Eqs. (3.50-3.51). The distribution of $u(x,y)$ above behaves as a flat plane as shown in Fig. 6.8.

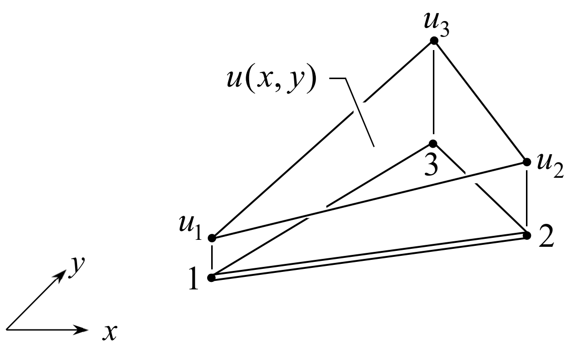


Figure 6.8 Distribution of $u(x,y)$ over the element.

Equation (6.11) can be written in the matrix form as,

$$u(x,y) = \underset{(1\times3)}{\lfloor N(x,y) \rfloor}\underset{(3\times1)}{\{u\}} \qquad (6.12)$$

The finite element equations can be derived by using the method of weighted residuals as explained in chapter 4. The method is applied to the differential equation, Eq. (6.9), to yield the finite element equations in the matrix form as,

$$\underset{(3\times3)}{[K]}\underset{(3\times1)}{\{u\}} + \underset{(3\times3)}{[H]}\underset{(3\times1)}{\{u\}} = \underset{(3\times1)}{\{Q\}} + \underset{(3\times1)}{\{F\}} \qquad (6.13)$$

where

$$\underset{(3\times3)}{[K]} = \int_{\Omega_e} c\left(\underset{(3\times1)}{\left\{\frac{\partial N}{\partial x}\right\}} \underset{(1\times3)}{\left\lfloor \frac{\partial N}{\partial x} \right\rfloor} + \underset{(3\times1)}{\left\{\frac{\partial N}{\partial y}\right\}} \underset{(1\times3)}{\left\lfloor \frac{\partial N}{\partial y} \right\rfloor} \right) d\Omega \qquad (6.14a)$$

$$\underset{(3\times3)}{[H]} = \int_{\Omega_e} a \underset{(3\times1)}{\{N\}} \underset{(1\times3)}{\lfloor N \rfloor} \, d\Omega \qquad (6.14b)$$

$$\underset{(3\times1)}{\{Q\}} = \int_{\Gamma_e} c \underset{(3\times1)}{\{N\}} \left(\frac{\partial u}{\partial x} n_x + \frac{\partial u}{\partial y} n_y \right) d\Gamma \qquad (6.14c)$$

$$\underset{(3\times1)}{\{F\}} = \int_{\Omega_e} f \underset{(3\times1)}{\{N\}}\, d\Omega \tag{6.14d}$$

The above element matrices which are in the integral form can be derived in the closed-form expressions, so that they can be implemented into the computer program directly. These element matrices in closed-form expressions are,

$$\underset{(3\times3)}{[K]} = \frac{c}{4A}\begin{bmatrix} b_1^2+c_1^2 & b_1b_2+c_1c_2 & b_1b_3+c_1c_3 \\ & b_2^2+c_2^2 & b_2b_3+c_2c_3 \\ \text{symmetric} & & b_3^2+c_3^2 \end{bmatrix} \tag{6.15a}$$

$$\underset{(3\times3)}{[H]} = \frac{a\,A}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \tag{6.15b}$$

and

$$\underset{(3\times1)}{\{F\}} = \frac{f\,A}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \tag{6.15c}$$

where b_i and c_i, $i = 1, 2, 3$ are given in Eq. (3.51). The vector $\{Q\}$ is replaced by the appropriate boundary conditions if the element is on the domain boundary. For an interior element, the vector $\{Q\}$ is cancelled with those from the surrounding elements. To understand the use of the finite element equations and their matrices above, we will first utilize the MATLAB PDE Toolbox to analyze a simple two-dimensional problem as shown in the section.

6.3.3 Example

Given the elliptic partial differential equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{6.16}$$

for $0 \le x \le 2$, $0 \le y \le 1$, with the boundary conditions as shown in Fig. 6.9. Employ the PDE Toolbox to obtain a finite element solution by creating a proper mesh. Compare the computed solution with the exact solution of,

$$\bar{u}(x, y) = \sin(\pi x/2)\sinh(\pi y/2)/\sinh(\pi/2) \qquad (6.17)$$

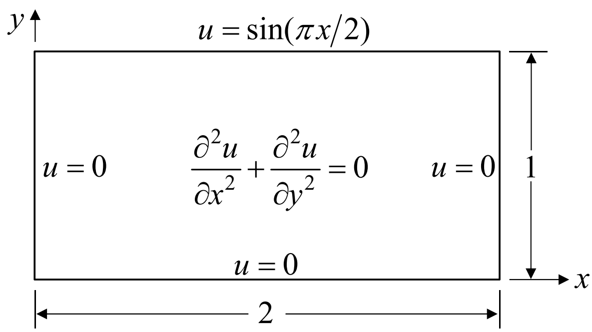


Figure 6.9 Two-dimensional elliptic problem statement.

The PDE Toolbox can be used to solve this problem conveniently. The rectangular domain is firstly created and a mesh consisting of triangular elements is then generated. The Dirichlet boundary conditions of $u = 0$ are imposed along the left, bottom and right edges. The condition along the top edge is applied by entering sin(*pi***x*/2) for *r* in the Boundary Condition box. The differential equation is specified by entering $c = 1$, $a = 0$ and $f = 0$ in the PDE Specification box under the commands of PDE > PDE Specification.

The generated finite element model consists of 672 elements and 367 nodes as shown in Fig. 6.10. The computed finite element solution is displayed in form of the color fringe plot in Fig. 6.11. The finite element solution is compared with the exact solution in form of the surface plot as shown in Fig. 6.12. The figures demonstrate that the toolbox can provide the finite element solution with high accuracy.

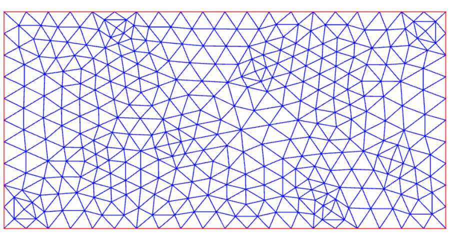


Figure 6.10 A finite element mesh with 672 triangles.

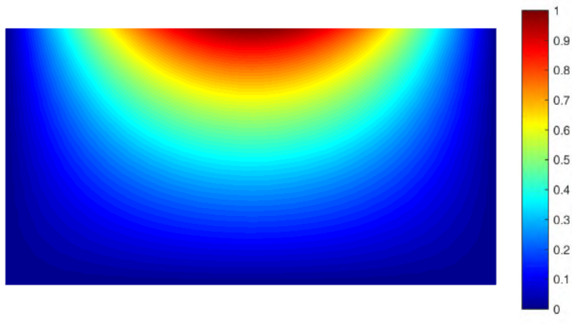


Figure 6.11 Fringe plot of the computed $u(x,y)$ distribution.

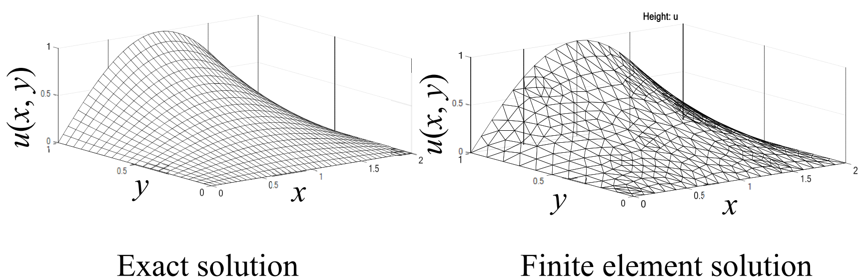


Figure 6.12 Comparative $u(x,y)$ distributions.

This example is also analyzed by using the FEATool software. The computed $u(x,y)$ distribution obtained from the software is displayed in form of the fringe and surface plots as shown in Figs. 6.13 and 6.14, respectively.

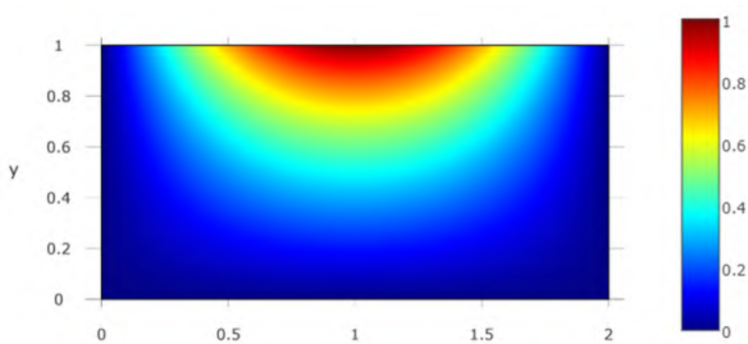


Figure 6.13 Fringe plot of the computed $u(x,y)$ distribution using FEATool.

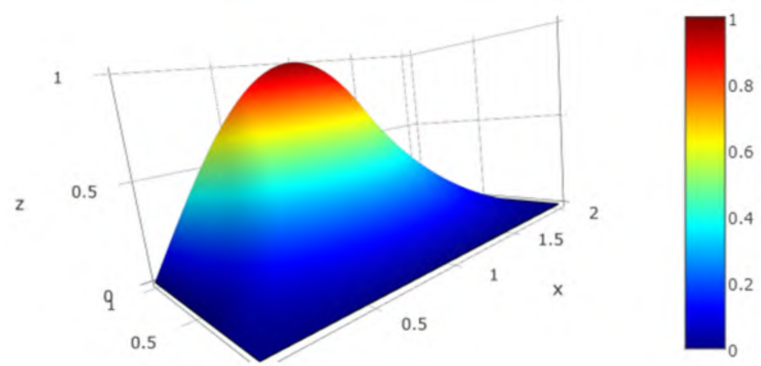


Figure 6.14 Surface plot of the computed $u(x,y)$ distribution using FEATool.

6.4 One-Dimensional Parabolic Problem

6.4.1 Differential Equation

The standard form of the parabolic partial differential equation in one dimension is,

$$d\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(c\frac{\partial u}{\partial x}\right) + a\,u \;=\; f \tag{6.18}$$

where d, c, a and f are constants or may be function of x. The basic unknown is the dependent variable $u(x,t)$ that depends on the x-coordinates and time t.

The parabolic partial differential equation above is solved together with the boundary conditions and an initial condition for a given domain length. The boundary conditions may be in form of the Dirichlet condition by specifying u or the Neumann condition by specifying $\partial u/\partial n$. An initial condition of $u(x,0)$ is required at time $t = 0$. Result of $u(x,t)$ represents the solution to the problem that varies with the x-coordinate and time t.

6.4.2 Element Equations and Matrices

The finite element equations corresponding to the parabolic differential equation can be derived by applying the method of weighted residuals. The application is similar to that presented in Chapter 4 leading to the finite element equations in the form,

$$[D]\{\dot{u}\} + [K]\{u\} + [H]\{u\} = \{Q\} + \{F\} \tag{6.19}$$

where

$$[D] = \int_{\Omega_e} d\{N\}\lfloor N \rfloor d\Omega \tag{6.20}$$

Details of the [K], [H] matrices and {Q}, {F} vectors are given in Eq. (6.5). The [D] matrix in Eq. (6.20) can be derived in closed-form expressions for simple element types. As an example, the [D] matrix for the two-node linear element is,

$$[D] = \frac{dL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{6.21}$$

To solve for the transient response, the finite element equations above are written as,

$$[D]\{\dot{u}\} + [\bar{K}]\{u\} = \{\bar{F}\} \tag{6.22}$$

where

$$[\bar{K}] = [K] + [H] \tag{6.23a}$$

and

$$\{\bar{F}\} = \{Q\} + \{F\} \tag{6.23b}$$

The rate of change of the unknown vector $\{\dot{u}\}$ may be determined by using the forward difference approximation as,

$$\{\dot{u}\} = \frac{\{u\}_{n+1} - \{u\}_n}{\Delta t} \tag{6.24}$$

where $\{u\}_{n+1}$ and $\{u\}_n$ are the solutions at the times t_{n+1} and t_n, respectively, while Δt denotes the time step. The finite element equations above then become,

$$\frac{1}{\Delta t}[D]\left(\{u\}_{n+1} - \{u\}_n\right) + [\bar{K}]\{u\}_n = \{\bar{F}\}_n$$

Or,

$$\frac{1}{\Delta t}[D]\{u\}_{n+1} = \frac{1}{\Delta t}[D]\{u\}_n - [\bar{K}]\{u\}_n + \{\bar{F}\}_n \tag{6.25}$$

It is noted that the terms on the right-hand side of Eq. (6.25) are all known at time t_n. Thus, the solution $\{u\}_{n+1}$ at the new time t_{n+1} can be determined directly. This method is called the Euler method which uses the forward difference approximation for $\dot{u}$ in form of the solutions u_{n+1} and u_n at times t_{n+1} and t_n, respectively.

If the rate of change of the unknown vector $\{\dot{u}\}$ in Eq. (6.24) is determined at the middle of the time step, the method is called the Crank-Nicolson method. Such method can provide more accurate solution as compared to the Euler method. However, the method requires a more computational time since a set of simultaneous equations is needed to be solved at each time step. More detail of the Crank-Nicolson method is provided in chapter 8

In summary, the finite element process for solving the parabolic equation is as follows. We begin from the parabolic differential equation where the unknown u is function of x and t. We apply the method of weighted residuals to transform the differential equation into the finite element equations. The finite element equations contain the unknowns of $\dot{u}$ and u. Next, we approximate $\dot{u}$ in form of u at the new time t_{n+1} and previous time t_n depending on the time marching scheme selected. Finally, the unknown of u_{n+1} at new time t_{n+1} can be determined.

6.4.3 Example

Given the parabolic partial differential equation in the form,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{6.26}$$

for $0 \leq x \leq \pi$ and $t > 0$. The boundary conditions are $u(0, t) = u(\pi, t) = 0$ and the initial condition is $u(x, 0) = \sin x + 2\sin(2x)$. Employ the FEATool software to solve for the finite element solution. Create a mesh by dividing the domain length into 10 elements. Compare the computed solution with the exact solution of,

$$\bar{u}(x,t) = e^{-t}\sin x + 2e^{-4t}\sin(2x) \tag{6.27}$$

at $t = 0$, 0.1 and 0.5.

To use FEATool to solve this problem, we follow the same procedure for analyzing the one-dimensional elliptic example as explained in section 6.2.3. The only differences are as follows. In the custom equation under the Equation Settings window, we enter `u' - (ux_x) = 0` to represent the governing parabolic differential equation of this problem. We also provide the initial condition by entering sin(x) + 2*sin(2*x) as u_0 under the Initial Conditions slot. Before solving for the transient response, we click at the Solve and Settings button, respectively, the Solver Settings window will appear. In this window, we select Time-Dependent under the Solver Type, and enter the desired Time step and Simulation time under Time Dependent Settings. After clicking at the Solve button, the transient response will be determined. A typical transient response of $u(x)$ at time $t = 0.1$ is shown in Fig. 6.15. The computed finite element solutions at various times are quite accurate as seen by comparing with the exact solution in Fig. 6.16.

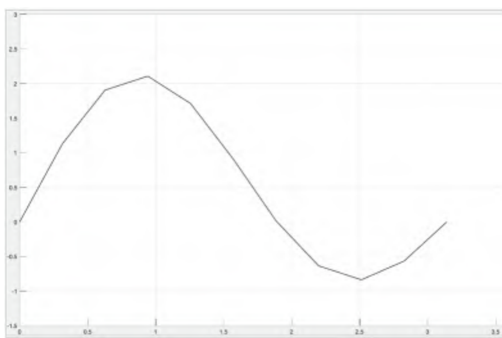


Figure 6.15 Finite element solution of $u(x)$ at time $t = 0.1$ from FEATool.

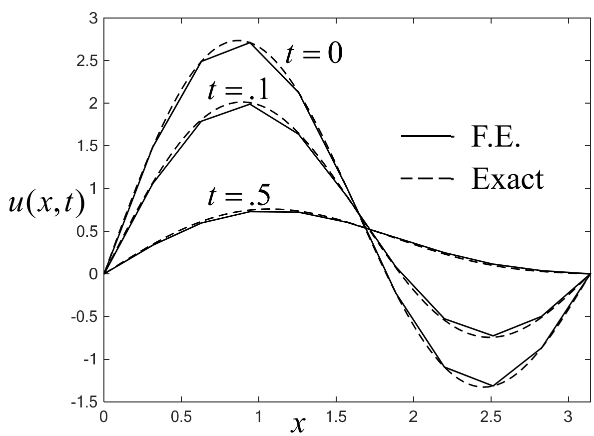


Figure 6.16 Comparative finite element and exact solutions at different times.

6.5 Two-Dimensional Parabolic Problem

6.5.1 Differential Equation

The standard form of the parabolic partial differential equation in two dimensions is,

$$d\frac{\partial u}{\partial t} - \left(\frac{\partial}{\partial x}\left(c\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(c\frac{\partial u}{\partial y}\right)\right) + a\,u \;=\; f \qquad (6.28)$$

where *d*, *c*, *a* and *f* are constants or may be function of *x* and *y*. The basic unknown is the dependent variable *u*(*x*, *y*, *t*) that depends on *x*-*y* coordinates and time *t*.

The parabolic partial differential equation above is solved together with the boundary conditions and an initial condition for a given domain geometry. The boundary conditions may be in form of the Dirichlet condition by specifying *u* or the Neumann condition by specifying $\partial u/\partial n$. An initial condition of *u*(*x*, *y*, 0) is required at time *t* = 0. Result of *u*(*x*, *y*, *t*) represents the solution to the problem that varies with the *x*-*y* coordinates and time *t*.

6.5.2 Element Equations and Matrices

The finite element equations corresponding to the parabolic differential equation can be derived by applying the method of weighted residuals. The application is similar to that presented earlier leading to the finite element equations as shown in Eq. (6.19). For the three-node triangular element, the [*D*] matrix is,

$$[D] \;=\; \frac{dA}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad (6.29)$$

To solve for the transient response, the same computational procedure as explained for the one-dimensional problem in section 6.5.1 can be used. Herein, we will utilize both of the MATLAB PDE and FEATool Toolboxes to solve a two-dimensional parabolic problem as demonstrated in the next section.

6.5.3 Example

Given the parabolic differential equation in the form,

$$\frac{\partial u}{\partial t} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0 \tag{6.30}$$

for $0 \leq x \leq 1$, $0 \leq y \leq 1$ and $t > 0$. The unit square domain is subjected to the boundary conditions of $u = 0$ along the four edges with the initial condition of,

$$u(x, y, 0) = x(1 - x^2)y(1 - y) \tag{6.31}$$

Employ the PDE Toolbox with an appropriate mesh to determine the transient solution response. Use the surface plot to compare the finite element solution at time $t = 0.02$ with the exact solution of,

$$\bar{u}(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{48(-1)^m ((-1)^n - 1)}{(m\, n\, \pi^2)^3} \sin(m\pi x) \sin(n\pi y) e^{-(m^2+n^2)\pi^2 t} \tag{6.32}$$

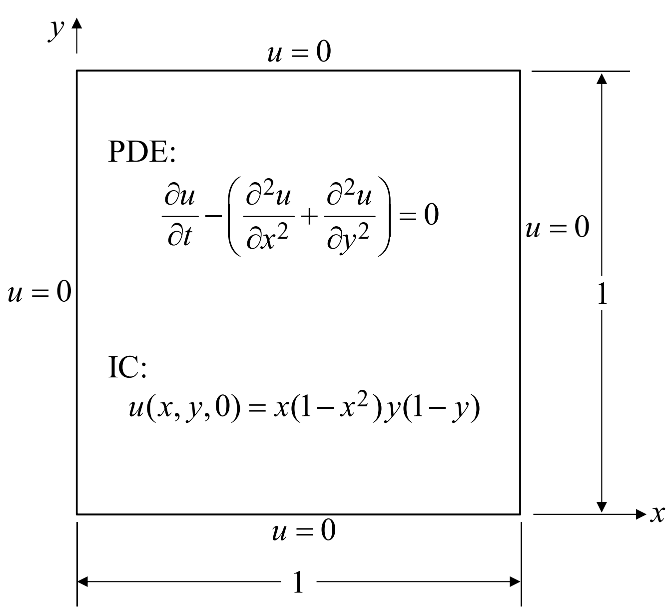


Figure 6.17 Two-dimensional parabolic problem statement.

The PDE Toolbox can be used to solve the parabolic differential equation for the transient solution of this problem easily. The unit square domain is first discretized into 1,248 triangular elements with 665 nodes as shown in Fig. 6.18. The Dirichlet boundary conditions of $u = 0$ are imposed along the four edges. The governing differential equation is specified by using the commands of PDE > PDE Specification. In the PDE Specification box, we select the Parabolic option and enter the values of $c = 1$, $a = 0$, $f = 0$, $d = 1$, and click OK.

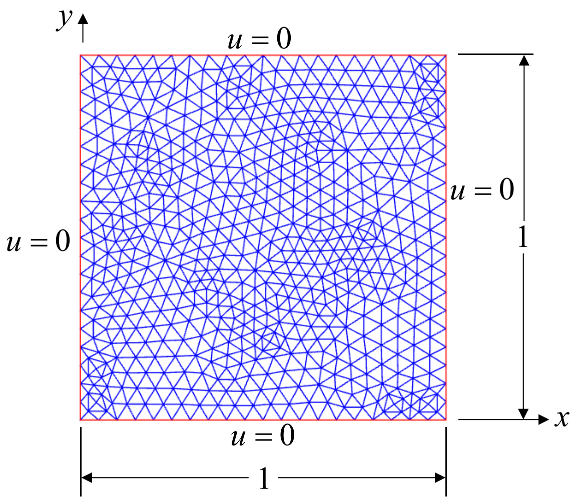
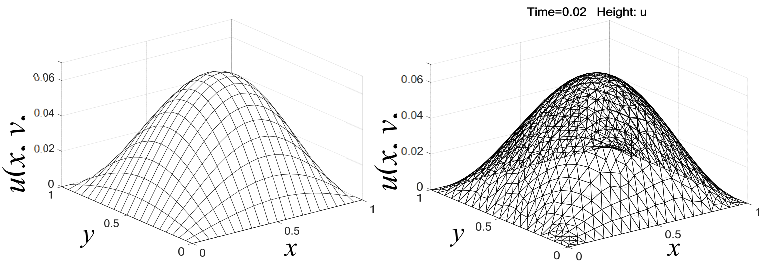


Figure 6.18 A finite element mesh with 1,248 triangles.

The initial condition is entered into the problem by using the commands of Solve > Parameters. In the Solve Parameters box, we enter 0:.005:.02 for Time: and *x*.*(1.–*x*.^2)**y*.*(1.–*y*) for *u*(*t*0):, and click OK. The computed solution is compared with the exact solution at time $t = .02$ as shown by the surface plots in Fig. 6.19. The figure demonstrates that the toolbox can provide accurate transient solution to the problem.

This example is also analyzed by using the FEATool software. The computed $u(x,y)$ distribution at time $t = 0.02$ obtained from the software is displayed in form of the fringe and surface plots as shown in Figs. 6.20 and 6.21, respectively.



Exact solution Approximate solution

Figure 6.19 Comparative $u(x,y)$ distributions at $t = 0.02$.

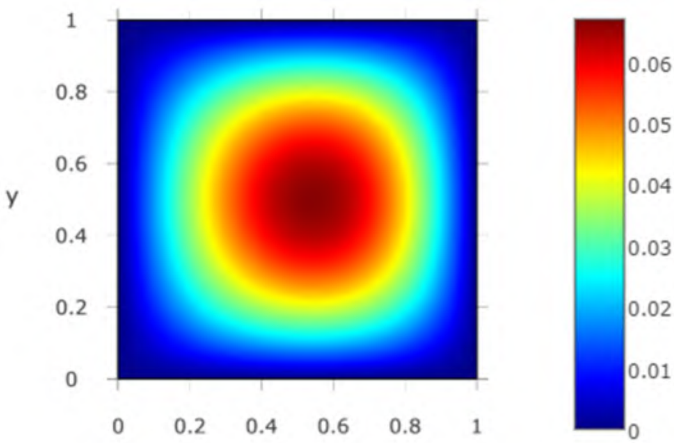


Figure 6.20 Fringe plot of the computed $u(x,y)$ distribution at $t = 0.02$ using FEATool.

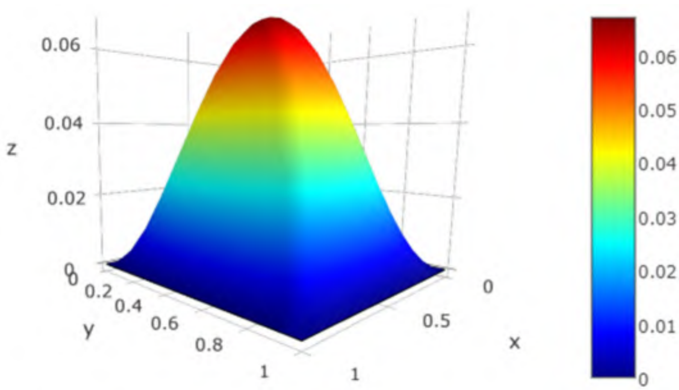


Figure 6.21 Surface plot of the computed $u(x,y)$ distribution at $t = 0.02$ using FEATool.

6.6 One-Dimensional Hyperbolic Problem

6.6.1 Differential Equation

The standard form of the hyperbolic partial differential equation in one dimension is,

$$d\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(c\frac{\partial u}{\partial x}\right) + a\,u \;=\; f \tag{6.33}$$

where d, c, a and f are constants or may be function of x. The basic unknown is the dependent variable $u(x,t)$ that depends on x-coordinate and time t.

The hyperbolic partial differential equation above is solved together with the boundary and initial conditions for a given domain length. The boundary conditions may be in form of the Dirichlet condition by specifying u or the Neumann condition by specifying $\partial u/\partial n$. The initial conditions of $u(x,0)$ and $\partial u/\partial t(x,0)$ are required at time $t = 0$. Result of $u(x,t)$ represents the solution to the problem that varies with the x-coordinate and time t.

6.6.2 Element Equations and Matrices

The finite element equations corresponding to the hyperbolic differential equation can be derived by applying the method of weighted residuals. The application is similar to that presented in Chapter 4 leading to the finite element equations in the form,

$$[D]\{\ddot{u}\} + [K]\{u\} + [H]\{u\} \;=\; \{Q\} + \{F\} \tag{6.34}$$

where

$$[D] \;=\; \int_{\Omega_e} d\{N\}\lfloor N\rfloor\, d\Omega \tag{6.35}$$

Details of the $[K]$, $[H]$ matrices and $\{Q\}$, $\{F\}$ vectors are given in Eqs. (6.4-6.5). The $[D]$ matrix occurred in the hyperbolic equation can be derived in closed-form expressions for simple element types. As an example, the $[D]$ matrix for the two-node linear element is the same as that shown in Eq. (6.21).

The finite element equations, Eq. (6.34), above can be written in short as,

$$[D]\{\ddot{u}\} + [\bar{K}]\{u\} \;=\; \{\bar{F}\} \tag{6.36}$$

where $$[\bar{K}] \quad = \quad [K] + [H] \tag{6.37a}$$

and $$\{\bar{F}\} \quad = \quad \{Q\} + \{F\} \tag{6.37b}$$

These finite element equations are in form of the second-order ordinary differential equations which depend on time *t*.

There are several ways to approximate the second-order derivative of *u* with rest to time, $\ddot{u}$, in form of *u* at the new time step *n*+1 and the previous time step *n*. The forward and central finite different approximations are normally used. The known nodal quantities at time step *n* are used to determine their new quantities at time step *n*+1 by using the time step interval of Δt. Details of such computational methods are explained in chapter 8.

6.6.3 Example

Given the hyperbolic partial differential equation in the form,

$$\frac{\partial^2 u}{\partial t^2} \quad = \quad \frac{\partial^2 u}{\partial x^2} \tag{6.38}$$

for $0 \leq x \leq 1$ and $t > 0$. The boundary conditions are $u(0, t) = u(1, t) = 0$ and the initial condition are,

$$u(x,0) \quad = \quad \sin(2\pi x) \qquad ; \qquad \frac{\partial u}{\partial t}(x,0) \quad = \quad 2\pi \sin(2\pi x)$$

Employ the FEATool software to solve for the finite element solution. Divide the domain length into 10 equal intervals. Plot to compare the computed solution with the exact solution of,

$$\bar{u}(x,t) \quad = \quad \sin(2\pi x)[\sin(2\pi t) + \cos(2\pi t)] \tag{6.39}$$

at $t = 0$, 0.20 and 0.45.

To utilize the FEATool software to solve this problem, we follow the same procedure for analyzing the one-dimensional elliptic example as explained in section 6.2.3. The only differences are as follows. We need to separate the second-order time-derivative differential equation, representing the governing hyperbolic differential equation of this problem, into two first-order time-derivative equations. In the custom equation under the Equation Settings window, we enter u' – u2_t = 0 and u2' – (ux_x) = 0 under the `Custom Equation` and `Custom Equation 2` (ce and ce2 buttons), respectively. The two initial conditions are specified by entering `sin(2*pi*x)` and `2*pi*sin(2*pi*x)` as u_0 and $u2_0$ under the Initial Conditions slot. Before solving for the transient response, we click at the Solve and Settings button, respectively, the Solver Settings window will appear. In this window, we select Time-Dependent under the Solver Type, and enter the desired Time step and Simulation time under Time Dependent Settings. After clicking at the Solve button, the transient response will be determined. A typical transient response of $u(x)$ at time $t = 0.2$ is shown in Fig. 6.22. The computed finite element solutions at various times are quite accurate as seen by comparing with the exact solution in Fig. 6.23.

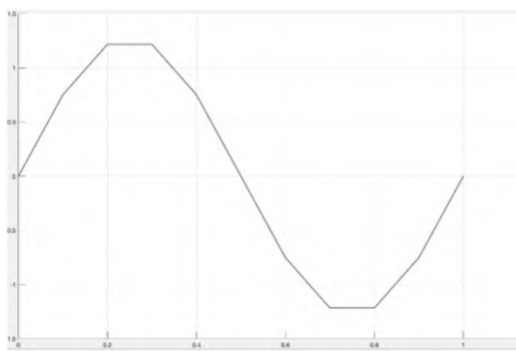


Figure 6.22 Finite element solution of $u(x)$ at time $t = 0.2$ from FEATool.

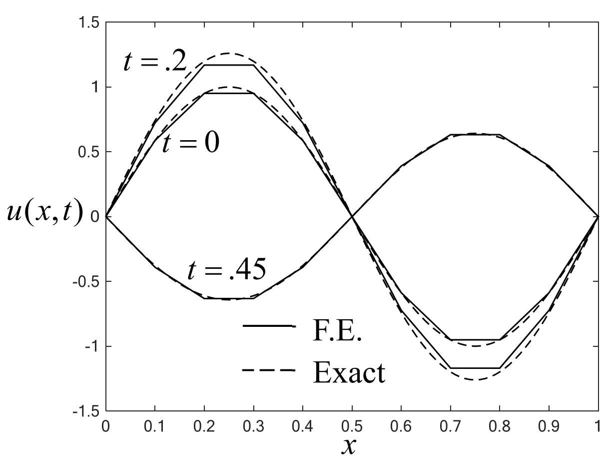


Figure 6.23 Comparative finite element and exact solutions at different times.

6.7 Two-Dimensional Hyperbolic Problem

6.7.1 Differential Equation

The standard form of the hyperbolic partial differential equation in two dimensions is,

$$d\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial}{\partial x}\left(c\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(c\frac{\partial u}{\partial y}\right)\right) + a\,u \quad = \quad f \qquad (6.40)$$

where d, c, a and f are constants or may be function of x and y. The basic unknown is the dependent variable $u(x, y, t)$ that depends on x-y coordinates and time t.

The hyperbolic partial differential equation above is solved together with the boundary and initial conditions for a given domain geometry. The boundary conditions may be in form of the Dirichlet condition by specifying u or the Neumann condition by specifying $\partial u/\partial n$. The initial conditions of $u(x, y, 0)$ and $\partial u/\partial t$ $(x, y, 0)$ are required at time $t = 0$. Result of $u(x, y, t)$ represents the solution to the problem that varies with the x-y coordinates and time t.

6.7.2 Element Equations and Matrices

Similar to the one-dimensional problem, the derived finite element equations are in the form,

$$[D]\{\ddot{u}\} + [K]\{u\} + [H]\{u\} \quad = \quad \{Q\} + \{F\} \qquad (6.41)$$

For the three-node triangular element, the element matrices of $[D]$, $[K]$, $[H]$, $\{Q\}$ and $\{F\}$ are given in section 6.5.2 and 6.6.2. Again the finite element equations above, Eq. (6.41), can be written in short as,

$$[D]\{\ddot{u}\} + \left[\bar{K}\right]\{u\} \quad = \quad \left\{\bar{F}\right\} \qquad (6.42)$$

where $$\left[\bar{K}\right] \quad = \quad [K] + [H] \qquad (6.43a)$$

and $$\left\{\bar{F}\right\} \quad = \quad \{Q\} + \{F\} \qquad (6.43b)$$

These finite element equations are in form of the second-order ordinary differential equations which depend on time t. The forward and central finite different approximations are normally used to represent the second-order derivative of u with respect to time t. Accuracy of the finite element method for solving two-dimensional problems can be measured by using problems that have exact solutions. Very few hyperbolic problems in two dimensions have their exact solutions. Exact solutions are available for problems with simple geometry as the one shown in the following example.

6.7.3 Example

Given the hyperbolic partial differential equation in the form,

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0 \qquad (6.44)$$

for a unit square domain with the boundary conditions of $u = 0$ at any time t along the four edges as shown in the figure. The initial conditions are,

$$u(x, y, 0) = x(1-x)y(1-y) \quad \text{and} \quad \frac{\partial u}{\partial t}(x, y, 0) = 0$$

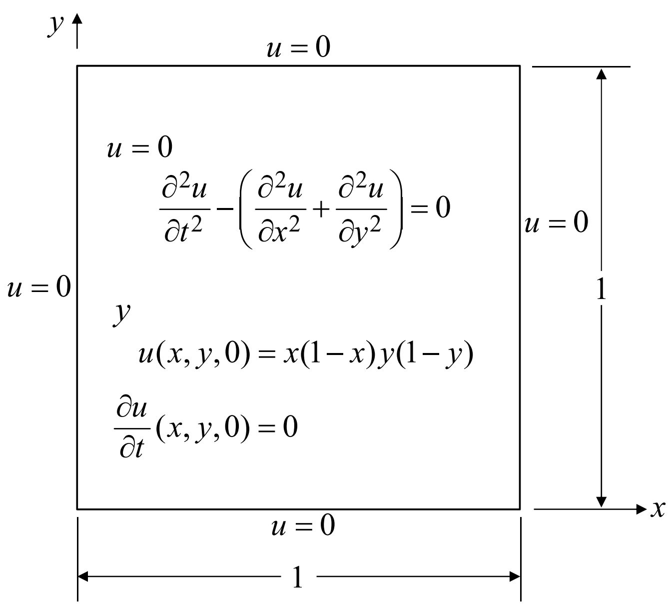


Figure 6.24 Two-dimensional hyperbolic problem statement.

Employ the MATLAB PDE and FEATool Toolboxes with an appropriate mesh to find the finite element solution at time $t = 0.35$. Use the surface plot to compare the computed solution with the exact solution of,

$$\bar{u}(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{16((-1)^m - 1)((-1)^n - 1)}{(m\,n\,\pi^2)^3} \cdot$$

$$\sin(m\pi x)\sin(n\pi y)\cos(\sqrt{(m^2+n^2)\pi^2}\,t) \quad (6.45)$$

In the MATLAB PDE Toolbox, the unit square domain can be created easily by using drawing tools through the commands of Draw > Rectangle/square. The Dirichlet boundary condition of u = 0 along the four edges is applied by using the commands of Boundary > Specify Boundary Conditions.

A mesh with 1,248 triangular elements and 665 nodes is generated as shown in Fig. 6.25. Coefficients of the governing hyperbolic equation are specified by using the commands of PDE > PDE Specification. In the PDE Specification box, we select the Hyperbolic button, enter the values of c = 1, a = 0, f = 0, d = 1 into the spaces provided and click OK.

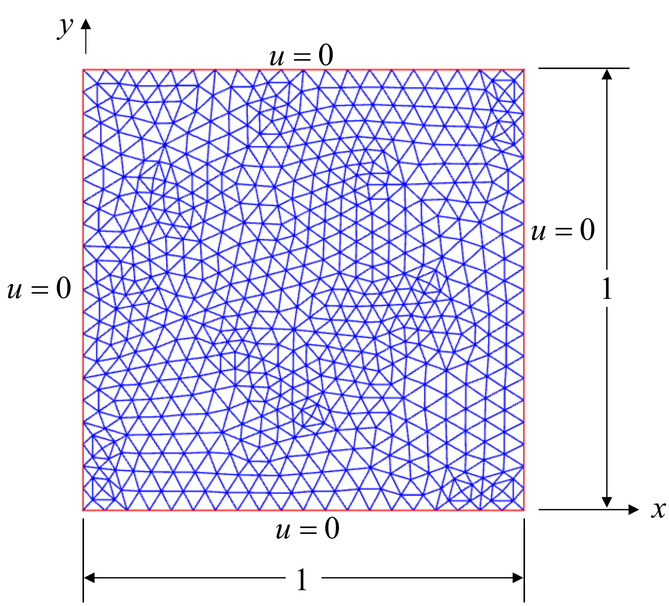
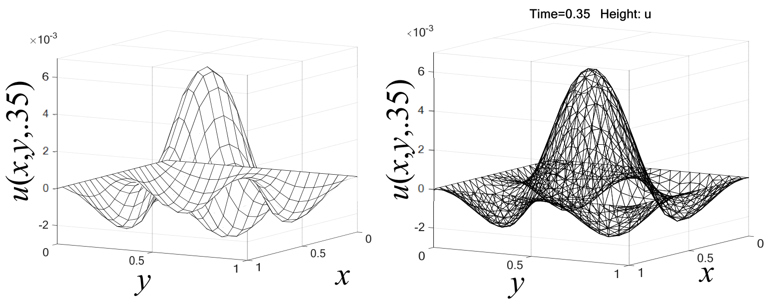


Figure 6.25 A finite element mesh with 1,248 triangles.

The initial conditions are specified by using the commands of Solve > Parameters. In the Solve Parameters box, we enter 0.:.05:.35 for Time:, *x*.*(1.–*x*).**y*.*(1.–*y*) for *u*(*t*0): and 0.0 for $u'(t0)$:. Then, we solve the problem by using the commands of Solve > Solve PDE. The computed solution is compared with the exact solution at time *t* = 0.35 by the surface plot as shown in Fig. 6.26. The comparison demonstrates that the toolbox can provide accurate solution as compared to the exact solution.



Exact solution Approximate solution

Figure 6.26 Comparative *u*(*x*,*y*) distributions at *t* = 0.35.

This example is also analyzed by using the FEATool Toolbox. The computed *u*(*x*,*y*) distribution at time *t* = 0.35 obtained from the software is displayed in form of the fringe and surface plots as shown in Figs. 6.27 and 6.28, respectively.

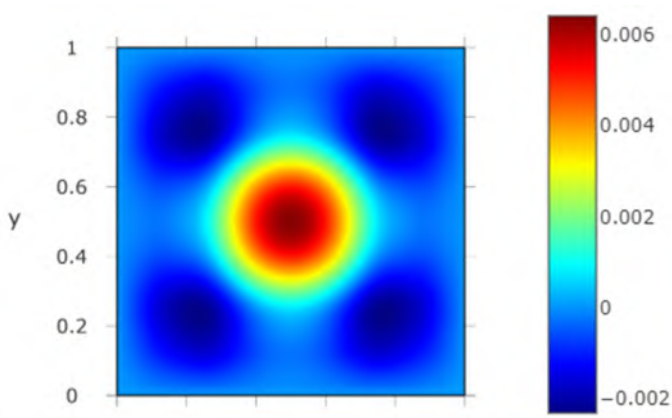


Figure 6.27 Fringe plot of the computed *u*(*x*,*y*) distribution at *t* = 0.35 using FEATool.

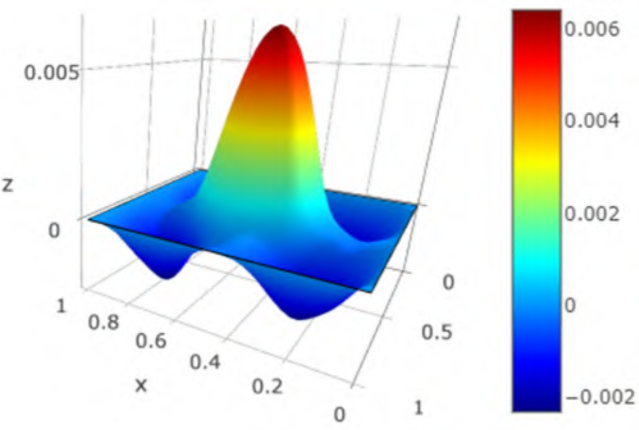


Figure 6.28 Surface plot of the computed $u(x,y)$ distribution at $t = 0.35$ using FEATool.

6.8 Closure

In this chapter, we have explored the use of finite element packages to solve three fundamental differential equations commonly encountered in scientific and engineering problems: elliptic, parabolic, and hyperbolic partial differential equations. To solve these equations, we employed two computer software toolboxes: the MATLAB PDE Toolbox and FEATool Multiphysics Toolbox.

We began the chapter by presenting the standard form of the elliptic, parabolic, and hyperbolic partial differential equations in two dimensions. We also discussed the possible boundary and initial conditions that may arise. To tackle these differential equations, we utilized both finite element toolboxes to solve problems in one and two dimensions. We started with one-dimensional problems, as they are more straightforward and serve as illustrative examples. Additionally, exact solutions for these problems are available, allowing for a comparison with the finite element solutions.

We further demonstrated the finite element analysis capabilities of the toolboxes using two-dimensional examples that have exact solutions. Both toolboxes offer a convenient way to discretize a given two-dimensional domain geometry into triangular elements. These examples showcase the versatility of the toolboxes in analyzing problems using the finite element method. They serve as an initial step before utilizing commercial finite element software to tackle more practical problems.

Exercises

1. Use the FEATool Toolbox to solve the non-homogeneous ordinary differential equation,

$$\frac{d^2u}{dx^2} = -e^x \qquad 0 \le x \le 1$$

with the boundary conditions of $u(0) = 0$ and $u(1) = 0$. Discretize the domain length into 2, 4 and 10 elements. Plot to compare the solutions with the exact solution of,

$$\bar{u}(x) = 1 - e^x + x(e-1)$$

2. Use the FEATool Toolbox to solve the non-homogeneous ordinary differential equation,

$$\frac{d^2u}{dx^2} + \frac{1}{4}u = 8 \qquad 0 \le x \le 10$$

with the boundary conditions of $u(0) = 0$ and $u(10) = 0$. Discretize the domain length into 2, 4 and 10 elements. Plot to compare the solutions with the exact solution of,

$$\bar{u}(x) = 32\left[\frac{\cos(5)-1}{\sin(5)}\sin\left(\frac{x}{2}\right) - \cos\left(\frac{x}{2}\right) + 1\right]$$

3. Use the FEATool Toolbox to solve the non-homogeneous ordinary differential equation,

$$\frac{d^2u}{dx^2} + 6u = 10x \qquad 0 \le x \le 2$$

with the boundary conditions of $u(0) = u(2) = 0$. Discretize the domain length into 2, 4 and 10 elements. Plot to compare the solutions with the exact solution of

$$\bar{u}(x) = \frac{5}{3}\left[x - \frac{2\sin\sqrt{6}x}{\sin(2\sqrt{6})}\right]$$

4. Use the FEATool Toolbox to solve the homogeneous ordinary differential equation,

$$\frac{d^2u}{dx^2} + 3\frac{du}{dx} + 2u = 0 \qquad 0 \le x \le 1$$

with the boundary conditions of $u(0) = 0$ and $u'(1) = 0$. Discretize the domain length into 2, 4 and 10 elements. Plot to compare the solutions with the exact solution of,

$$\bar{u}(x) = e^{-2x}(e - 2e^x)/(e-2)$$

5. Use the FEATool or PDE Toolbox to solve the elliptic differential equation in form of the Laplace equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad 0 \le x \le 1,\ 0 \le y \le 1$$

with the boundary conditions of,

$$u(x,0) = 0, \qquad u(x,1) = x \qquad 0 \le x \le 1$$

$$u(0,y) = 0, \qquad u(1,y) = y \qquad 0 \le y \le 1$$

Solve for a solution by using a mesh with proper element size. Employ the surface plot to compare the computed solution with the exact solution of,

$$\bar{u}(x,y) = xy$$

6. Use the FEATool or PDE Toolbox to solve the elliptic differential equation in form of the Poisson equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -1 \qquad 0 \le x \le 1,\ 0 \le y \le 1$$

with the boundary conditions of $u(x, y) = 0$ along the four edges. Employ the surface plot to compare the computed solution with the exact solution of,

$$\bar{u}(x,y) = \frac{16}{\pi^4}\sum_{\substack{j=1\\odd}}^{\infty}\sum_{\substack{k=1\\odd}}^{\infty}\frac{\sin(j\pi x)\sin(k\pi y)}{j^3k^2 + j^2k^3}$$

7. Use the FEATool or PDE Toolbox to solve the elliptic differential equation in form of the Poisson equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -2\left(x^2 + y^2\right) \qquad 0 \le x \le 1,\ 0 \le y \le 1$$

with the boundary conditions of,

$$u(x,0) = 1 - x^2, \quad u(x,1) = 2(1-x^2), \qquad 0 \le x \le 1$$

$$u(0,y) = 1 + y^2, \quad u(1,y) = 0, \qquad 0 \le y \le 1$$

Solve for a solution by using a mesh with proper element size. Employ the surface plot to compare the computed solution with the exact solution of,

$$\bar{u}(x,y) = (1 - x^2)(1 + y^2)$$

8. Use the FEATool or PDE Toolbox to solve the elliptic differential equation in form of the Poisson equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (x^2 + y^2)e^{xy} \qquad 0 \le x \le 2,\ 0 \le y \le 1$$

with the boundary conditions of,

$$u(x,0) = 1, \quad u(x,1) = e^{x} \qquad 0 \le x \le 2$$

$$u(0,y) = 1, \quad u(2,y) = e^{2y} \qquad 0 \le y \le 1$$

Solve for a solution by using a mesh with proper element size. Employ the surface plot to compare the computed solution with the exact solution of,

$$\bar{u}(x,y) = e^{xy}$$

9. Employ the FEATool Toolbox to solve the parabolic partial differential equation in one dimension,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le 1,\ t > 0$$

for $0 \le t \le 0.5$ with the boundary conditions of $u(0, t) = u(1, t) = 0$ and the initial condition of $u(x, 0) = \sin(\pi x)$. Discretize the domain length into 10 elements. Plot to compare the finite element solution at appropriate times with the exact solution of,

$$\bar{u}(x, t) = e^{-\pi^2 t} \sin(\pi x)$$

10. Employ the FEATool Toolbox to solve the parabolic partial differential equation in one dimension,

$$\frac{\partial u}{\partial t} = \frac{1}{\pi^2}\frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le 1,\ t > 0$$

for $0 \le t \le 1$ with the boundary conditions of $u(0, t) = u(1, t) = 0$ and the initial condition of $u(x, 0) = \cos(\pi(x-0.5))$. Discretize the domain length into 10 elements. Plot to compare the finite element solution at appropriate times with the exact solution of,

$$\bar{u}(x, t) = e^{-\pi^2 t} \cos(\pi(x - 0.5))$$

11. Employ the FEATool Toolbox to solve the parabolic partial differential equation in one dimension,

$$\frac{\partial u}{\partial t} = 5\frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le 1,\ t > 0$$

for $0 \le t \le 0.05$ with the boundary conditions of $u(0, t) = u(1, t) = 0$ and the initial condition of $u(x, 0) = x(1-x^2)$. Discretize the domain length into 10 elements. Plot to compare the finite element solution at appropriate times with the exact solution of,

$$\bar{u}(x, t) = -\frac{12}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3} e^{-5n^2\pi^2 t} \sin(n\pi x)$$

12. Employ the FEATool Toolbox to solve the parabolic partial differential equation in one dimension,

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2} \;=\; 2 \qquad\qquad 0\leq x\leq 1,\; t>0$$

for $0 \leq t \leq 1$ with the boundary conditions of $u(0, t) = u(1, t) = 0$ and the initial condition of $u(x, 0) = \sin\pi x + x(1-x)$. Discretize the domain length into 10 elements. Plot to compare the finite element solution at appropriate times with the exact solution of,

$$\bar{u}(x,t) \;=\; e^{-\pi^2 t}\sin\pi x \;+\; x(1-x)$$

13. Employ the FEATool or PDE Toolbox to solve the two-dimensional problem governed by the parabolic partial differential equation,

$$\frac{\partial u}{\partial t}-\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \;=\; 0 \qquad\qquad t>0$$

For a square domain of $0 \leq x \leq 1$ and $0 \leq y \leq 1$ with the boundary conditions of,

$$\begin{aligned} u(0,y,t) &= 1, & u(1,y,t) &= 0 & 0\leq y\leq 1 \\ \frac{\partial u}{\partial y}(x,0,t) &= 0, & \frac{\partial u}{\partial y}(x,1,t) &= 0 & 0\leq x\leq 1 \end{aligned}$$

and the initial condition of,

$$u(x,y,0) \;=\; 1-x-\frac{1}{\pi}(\sin 2\pi x)$$

Discretize the domain into a mesh with proper element sizes. Plot to compare the finite element solutions at time $t = 0.01, 0.02$ and 0.05 with the exact solution of,

$$\bar{u}(x,y,t) \;=\; 1-x-\frac{1}{\pi}e^{-4\pi^2 t}\sin(2\pi x)$$

14. Employ the FEATool or PDE Toolbox to solve the two-dimensional problem governed by the parabolic partial differential equation,

$$\frac{\partial u}{\partial t} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 2 \qquad t > 0$$

for a rectangular domain of $0 \le x \le 1$ and $0 \le y \le 0.5$ with the boundary conditions of,

$$\begin{aligned} u(0, y, t) &= u(1, y, t) = 0 && 0 \le y \le 0.5 \\ \frac{\partial u}{\partial y}(x, 0, t) &= \frac{\partial u}{\partial y}(x, 0.5, t) = 0 && 0 \le x \le 1 \end{aligned}$$

and the initial condition of,

$$u(x, y, 0) = \sin(\pi x) + x(1 - x)$$

Discretize the domain into a mesh with proper element sizes. Plot to compare the finite element solutions at time $t = 0.1$ with the exact solution of,

$$\bar{u}(x, y, t) = e^{-\pi^2 t} \sin(\pi x) + x(1 - x)$$

Then, study the solution behavior that varies with time by creating an animated video.

15. Employ the FEATool or PDE Toolbox to solve the two-dimensional problem governed by the parabolic partial differential equation,

$$\frac{\partial u}{\partial t} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 2 \qquad t > 0$$

for a rectangular domain of $-1 \le x \le 1$ and $0 \le y \le 1$ with the boundary conditions of,

$$\begin{aligned} u(-1, y, t) &= u(1, y, t) = 0 && 0 \le y \le 1 \\ \frac{\partial u}{\partial y}(x, 0, t) &= \frac{\partial u}{\partial y}(x, 1, t) = 0 && -1 \le x \le 1 \end{aligned}$$

and the initial condition of,

$$u(x, y, 0) = 0$$

Discretize the domain into a mesh with proper element sizes. Plot to compare the finite element solutions at time $t = 0.1$, 0.2 and 0.5 with the exact solution of,

$$\bar{u}(x, y, t) = 1 - x^2 - \frac{32}{\pi^3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \left(\cos(2n+1)\frac{\pi x}{2} \right) e^{-(2n+1)^2 \pi^2 t/4}$$

16. Employ the FEATool or PDE Toolbox to solve the two-dimensional problem governed by the parabolic partial differential equation,

$$\frac{\partial u}{\partial t} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0$$

for a rectangular domain of $0 \le x \le \pi$ and $0 \le y \le 0.3$ with the Neumann boundary conditions of $\partial u / \partial n = 0$ along the four edges. The initial condition is,

$$u(x, y, 0) = \cos x$$

Discretize the domain into a mesh with proper element sizes. Plot to compare the finite element solutions at time $t = 0.2$ with the exact solution of,

$$u(x, y, t) = e^{-t} \cos x$$

17. Use the FEATool Toolbox to solve the hyperbolic partial differential equation,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le 1,\ t > 0$$

for $0 \le t \le 5$ with the boundary conditions of $u(0, t) = u(1, t) = 0$ and the initial conditions of $u(x, 0) = \sin(\pi x)$ and $\partial u / \partial t(x, 0) = 0$. Create a mesh by dividing the domain length into 10 elements. Plot the computed solutions at some selected times with the exact solution of,

$$\bar{u}(x, t) = \sin(\pi x) \cos(\pi t)$$

18. Use the FEATool Toolbox to solve the hyperbolic partial differential equation,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le \pi\,, \quad t > 0$$

for $0 \le t \le 4$ with the boundary conditions of $u(0, t) = u(\pi, t) = 0$ and the initial conditions of $u(x, 0) = x\cos(5x/2)$ and $\partial u/\partial t(x, 0) = 0$. Create a mesh by dividing the domain length into 10 elements. Plot the computed solutions at some selected times with the exact solution of,

$$\bar{u}(x,t) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}}{(5+2n)^2(5-2n)^2}\sin(nx)\cos(nt)$$

19. Use the FEATool Toolbox to solve the hyperbolic partial differential equation,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + x^2 + 5x + 3 \qquad 0 \le x \le 1\,, \; t > 0$$

for $0 \le t \le 4$ with the boundary conditions of $u(0, t) = u(1, t) = 0$ and the initial conditions of $u(x, 0) = 2x(1-x)$ and $\partial u/\partial t(x, 0) = 2\pi\sin(2\pi x)$. Create three meshes by dividing the domain length into 5, 10, 20 elements. Plot the computed solutions at some selected times with the exact solution of,

$$\bar{u}(x,t) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}}{(5+2n)^2(5-2n)^2}\sin(nx)\cos(nt)$$

20. Use the FEATool Toolbox to solve the hyperbolic partial differential equation,

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 2e^{-t}\sin x \qquad 0 \le x \le \pi\,, \; t > 0$$

for $0 \le t \le 1$ with the boundary conditions of $u(0, t) = u(\pi, t) = 0$ and the initial conditions of $u(x, 0) = \sin x$ and $\partial u/\partial t(x, 0) = -\sin x$. Create three meshes by dividing the domain length into

10, 20, 40 elements. Plot the computed solutions at some selected times with the exact solution of,

$$\bar{u}(x,t) = e^{-t}\sin x$$

21. Use the FEATool or PDE Toolbox to solve the hyperbolic partial differential equation,

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0 \qquad t > 0$$

for the rectangular domain of $0 \le x \le 1$ and $0 \le y \le 0.2$ with the boundary conditions of,

$$u(0,y,t) = u(1,y,t) = 0 \qquad 0 \le y \le 0.2$$
$$\frac{\partial u}{\partial y}(x,0,t) = \frac{\partial u}{\partial y}(x,.2,t) = 0 \qquad 0 \le x \le 1$$

and the initial conditions of,

$$u(x,y,0) = \sin(2\pi x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,y,0) = 2\pi\sin(2\pi x)$$

Create a mesh with proper element size. Use the surface plot to compare the computed solutions at $t = 0.3$ and 0.6 with the exact solution of,

$$\bar{u}(x,y,t) = \sin(2\pi x)[\sin(2\pi t) + \cos(2\pi t)]$$

22. Use the FEATool or PDE Toolbox to solve the hyperbolic partial differential equation,

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 2e^{-t}\sin x \qquad t > 0$$

for the rectangular domain of $0 \le x \le \pi$ and $0 \le y \le 0.3$ with the boundary conditions of,

$$u(0,y,t) = u(\pi,y,t) = 0 \qquad 0 \le y \le 0.3$$
$$\frac{\partial u}{\partial y}(x,0,t) = \frac{\partial u}{\partial y}(x,.3,t) = 0 \qquad 0 \le x \le \pi$$

and the initial conditions of,

$$u(x, y, 0) \quad = \quad \sin x \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x, y, 0) \quad = \quad -\sin x$$

Create a mesh with proper element size. Use the surface plot to compare the computed solutions at $t = 0.2$ and 0.5 with the exact solution of,

$$\bar{u}(x, y, t) \quad = \quad e^{-t} \sin x$$

23. Use the FEATool or PDE Toolbox to solve the hyperbolic partial differential equation,

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad = \quad x^2 + y^2 \qquad\qquad t > 0$$

for a unit square domain of $0 \leq x \leq 1$ and $0 \leq y \leq 1$ with the boundary conditions of $u = 0$ along the four edges. The initial conditions are,

$$u(x, y, 0) \quad = \quad x(1-x)\, y(1-y) \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x, y, 0) \quad = \quad 0$$

Create three meshes with the approximate element sizes of 0.5, 0.02 and 0.01. Study and provide comments on the convergence of solutions when the mesh is refined.

24. Use the FEATool or PDE Toolbox to solve the hyperbolic partial differential equation,

$$3\frac{\partial^2 u}{\partial t^2} - 5\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + 2u \quad = \quad 7 \qquad\qquad t > 0$$

for a rectangular domain of $0 \leq x \leq 6$ and $0 \leq y \leq 4$ with the boundary conditions of $u = 0$ along the four edges. The initial conditions are,

$$u(x, y, 0) \quad = \quad 0 \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x, y, 0) \quad = \quad 0$$

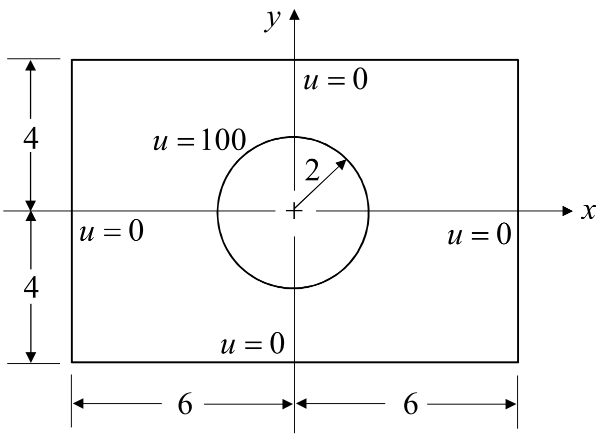
Create a mesh with proper element size. Use the surface plots to display the computed solutions at $t = 0.3$, 0.5 and 1.0.

Appendix B

FEATool Multiphysics Software

FEATool Multiphysics software can be downloaded and installed into MATLAB directly. FEATool is a finite element software capable to analyze many engineering applications. It is an easy-to-use software with Graphic User Interface (GUI) so that users can solve problems interactively on computer screen. The software has been developed to solve many forms of the differential equations for one-, two- and three-dimensional arbitrary geometry. Many differential equations such as the Poisson, conduction and diffusion equations, including custom equations provided by users can be solved conveniently. The software also contains built-in solvers for analyzing specific applications such as the electrostatics, magnetostatics, heat transfer, plane stress, plane strain, low-speed incompressible flow through the Navier-Stokes equations, and the Euler equations for high-speed inviscid compressible flow solutions.

In this appendix, we will use the FEATool multiphysics software to solve an elliptic equation on a rectangular domain with a circular cutout. The domain geometry and boundary conditions are shown in the figure.

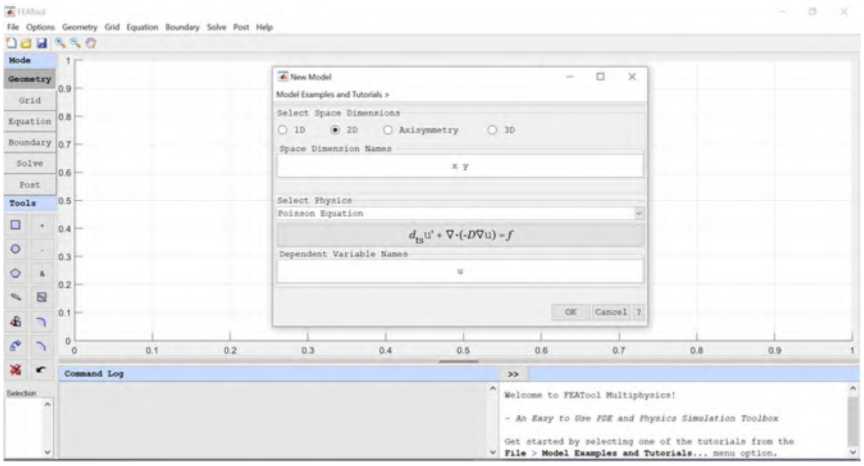


The problem is governed by the Laplace equation in the form,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

with the boundary conditions of $u = 100$ along the cutout inner edge and $u = 0$ along the four outer edges.

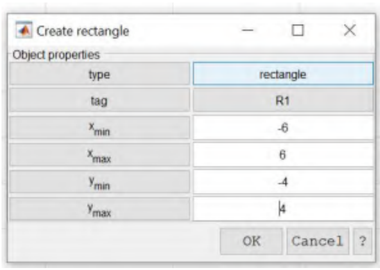
To employ the FEATool to analyze this problem, we first click the **FEATool Multiphysics** icon in the upper row beneath the **APPS** menu of MATLAB, the FEATool window will appear. Note that the Poisson Equation in two dimensions is the default option. If other type of problem physics is to be analyzed, click the **Cancel** button before proceeding. Herein, the given problem is governed by the Laplace equation which is a subset of the Poisson equation, we thus click the **OK** button.

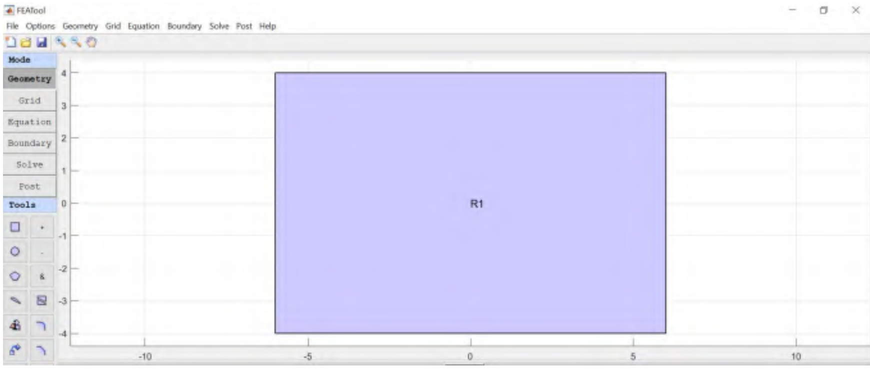


We will follow the finite element procedure according to the steps listed in the column under the **Mode** button on the left part of the screen. These steps consist of the Geometry, Grid, Equation, Boundary, Solve, and Post for post-processing the solution, respectively.

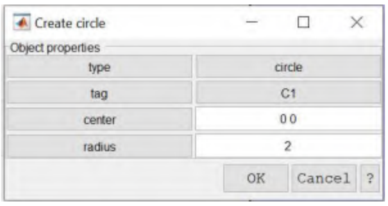
<u>Geometry</u>

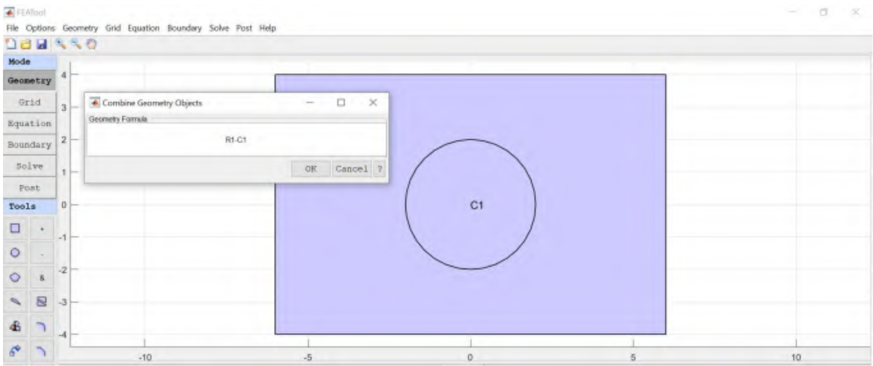
The very first step is to create a geometry with rectangle and a circle inside. We click at the **Geometry** command on the upper row of the window screen, a list of geometry construction commands appears. We select **Create Object…**, follow by **Rectangle**, a small **Create rectangle** window will pop up. We then enter x_{min}, x_{max}, y_{min}, y_{max} as **-6, 6, -4, 4**, respectively. After clicking the **OK** button, a rectangle denoted by **R1** appears.





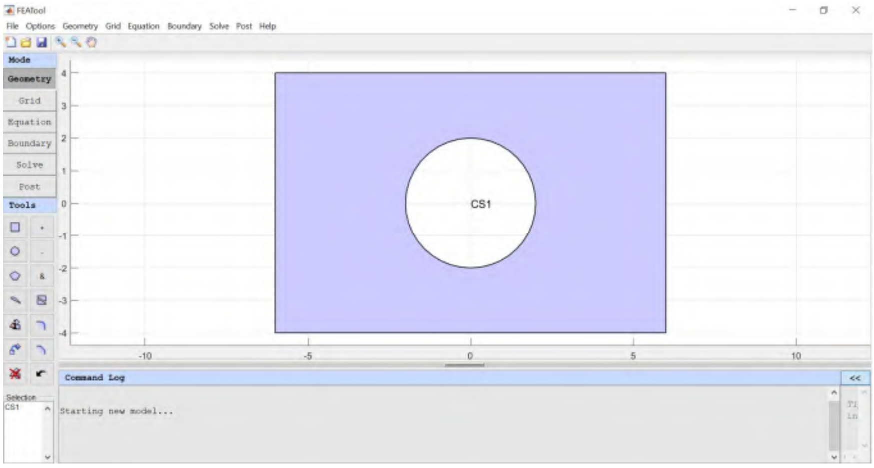
We then repeat the same process to create a circle at the center inside the rectangle. This is done by selecting **Geometry > Creating Object… > Circle**, a small **Create circle** window pops up. We enter **0 0** for the *x*- and *y*-coordinates of the circle center and **2** for the radius. After clicking the **OK** button, a circle denoted by **C1** appears.





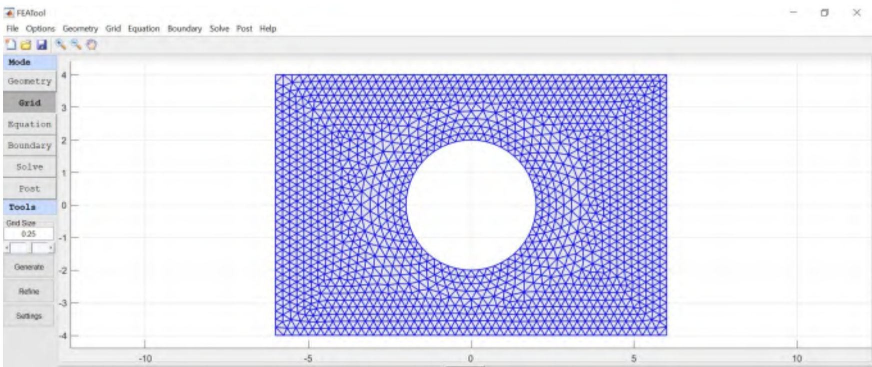
To subtract the circle from rectangle, we select **Geometry > Combine Objects…**, a small **Combine Geometry Objects** pops up. We then change the **Geometry Formula** from

R1+C1 to **R1-C1** and click the **OK** button, the desired domain of the rectangle with a circular hole is obtained.



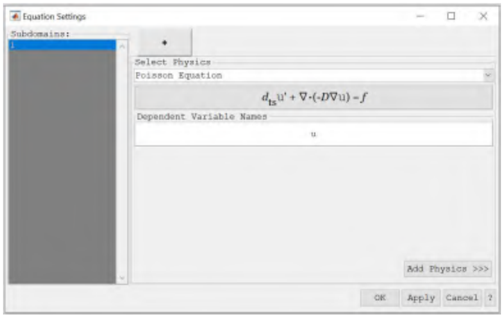
<u>Grid</u>

The next step is to create a mesh consisting of triangular or quadrilateral elements. Triangular elements are the default elements generated for two-dimensional domain. Once the **Grid** button is selected, a mesh consisting of triangular elements is generated automatically. After a desired element size is entered in the space under the **Grid Size** command, a new mesh consisting of all triangular elements will appear directly on the screen. The mesh as shown herein is generated by entering the element size of **0.25**. It is noted that, if the quadrilateral elements are preferred, they can be generated through the **Settings** option.

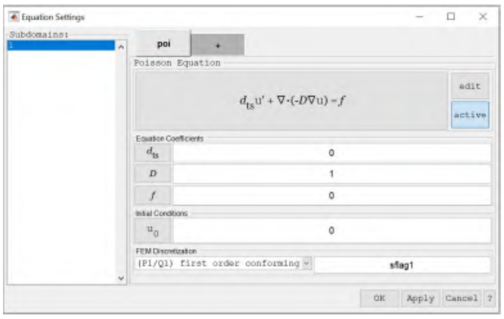


Equation

The differential equation that governs the problem can be now provided. We click at the **Equation** button, the **Equation Settings** window appears. We then select **Poisson Equation** option under **Select Physics**. Note that *u* is the dependent variable of the problem that varies with *x*-*y* coordinates and time *t*.

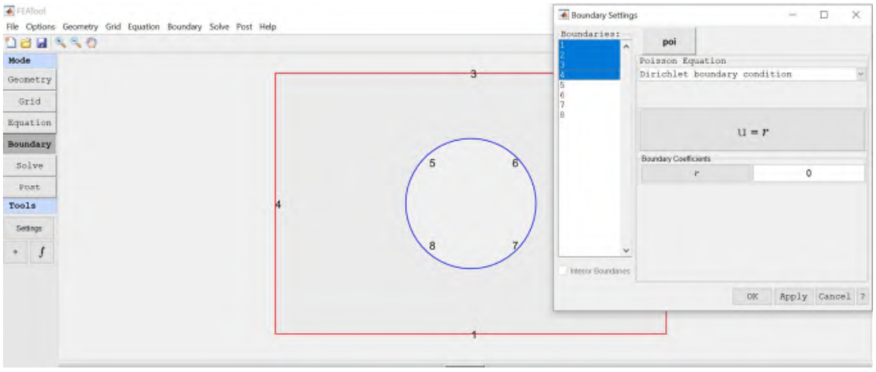


Next, we click at the **Add Physics** >>> button, details of the parameters in the differential equation will pop up. We enter d_{ts}, D, f as **0**, **1**, **0**, respectively, so that the differential equation agrees with the given Laplace equation of the problem, and click the **OK** button.

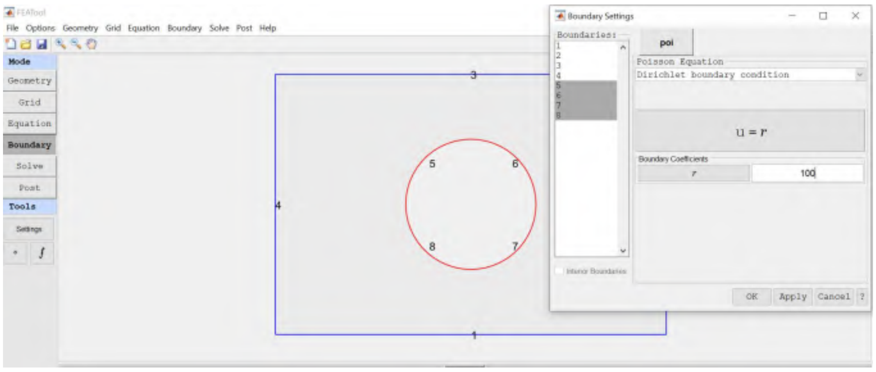


Boundary

The problem boundary conditions can be now applied by clicking at the **Boundary** button, the **Boundary Settings** window appears. After selecting **Boundaries 1**, **2**, **3**, **4**, their boundary colors will change from blue to red. We then enter the value of ***r*** as **0** according to the value of $u = 0$ which is specified along the outer four edges, and click the **OK** button.

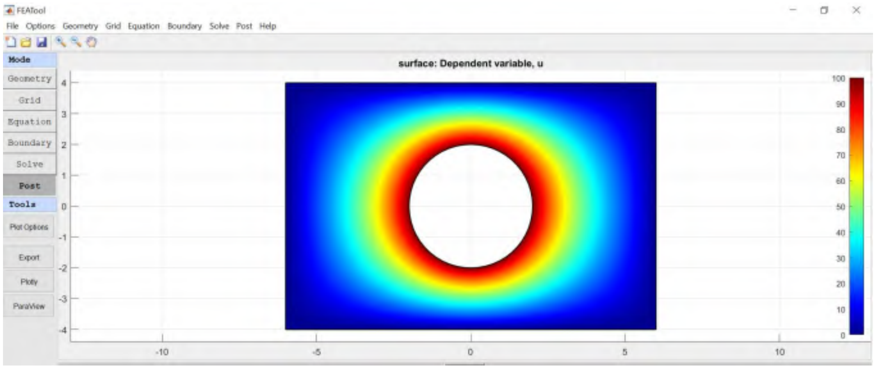


Next, we repeat the same process to apply the value of $u = 100$ along the circle edges. These edges are denoted by the **Boundaries 5**, **6**, **7**, **8**. This is done by entering the value of ***r*** as **100** and click the **OK** button.



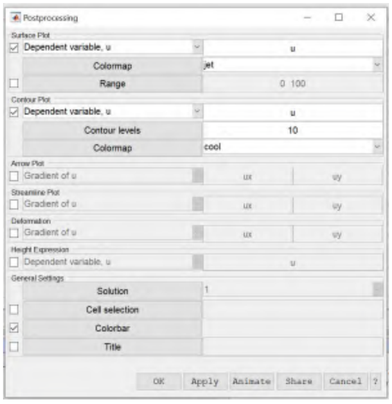
Solve

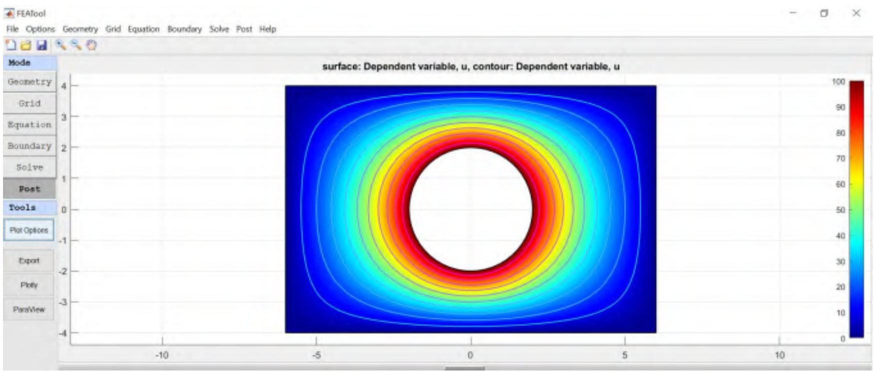
The problem is now ready to solve by clicking the **Solve** button, follow by the = button beneath the **Tool** button, the software will solve the problem and display the computed solution of u directly.

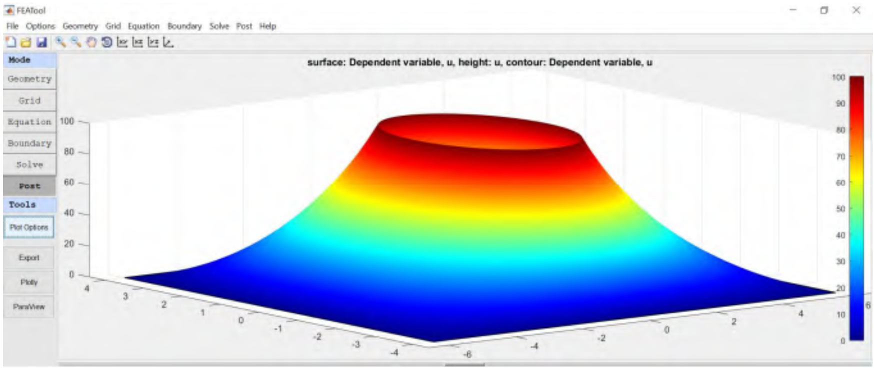


Post

There are several plotting options beneath the **Post** > **Tool** buttons. As examples, contour lines can be added into the fringe plot, or the height of the *u*-solution can be displayed in form of the three-dimensional plot.







For this problem, user can gain experience in using the FEATool multiphysics software by changing some parameters to obtain different output solutions. For examples, user may change the coefficients of the differential equation to $D = -3, f = 5$, or change the boundary conditions of the four outer edges to 25, 50, 75, 100. Different output solutions are obtained conveniently within a short time. The finite element method embedded in the FEATool can provide solutions to many types of differential equations subjected to different boundary conditions for one-, two- and three-dimensional models with arbitrary geometry.